The fluctuation function of the detrended fluctuation analysis - Investigation on the AR(1) process

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MAX-PLANCK-GESELLSCHAFT

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Range of correlations

- Time series: x_i with $i = 1, \ldots, N$
- Autocorrelation:

$$C(s) = \frac{\text{Cov}(x_i x_{i+s})}{\sqrt{\text{Var}(x_i)\text{Var}(x_{i+s})}} = \frac{\langle (x_i - \bar{x})(x_{i+s} - \bar{x}) \rangle}{\langle (x_i - \bar{x})^2 \rangle}$$

- Different classes of autocorrelation structures can be distinguished with respect to the form of their decay for large time lags s:
  - short-range correlations:
    - exponential decay

$$C(s) \sim \mathrm{e}^{-\frac{s}{s_c}}$$

  - long-range correlations:
    - algebraical decay

$$C(s) \sim s^{-\gamma} \text{ with } 0 < \gamma < 1$$

- Problem: Given data set ⇒ Determine $C(s)$ directly?
- Consider a stationary time series with a non-stationary trend
- Example: AR(1) process with linear trend

$$x_i = \underbrace{\sum_{k=2}^{N} a^{i-k}\eta_k}_{\text{AR(1) process}} + \underbrace{bi}_{trend}$$

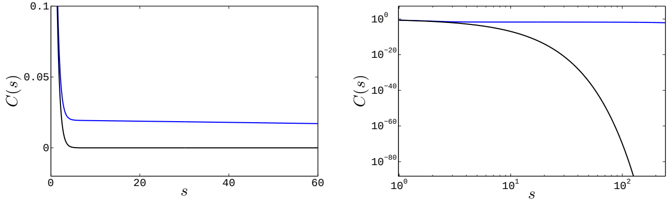


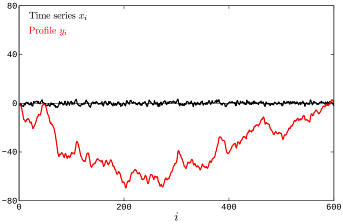
Figure : Stationary process (black) and with trend $b = 5 \cdot 10^{-4}$ (blue)

- Trend gives impression of long-range correlations and hides the true nature of the noise

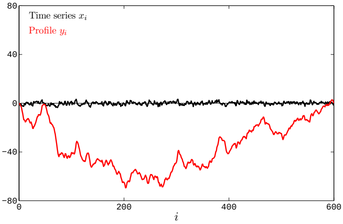
Detrended Fluctuation Analysis

- Estimating the autocorrelation function $C(s)$ from empirical data is effected by non-stationarities like trends and observational noise
- In 1994 Peng et al.[1] suggested Detrended Fluctuation Analysis (DFA) to indirectly gain information about the correlation structure imposed on a time series
- It has been applied to such diverse fields of interest as
  - DNA
  - human gait
  - heart rate dynamics
  - long-time weather records
  - economical time series
  - ...

[1] C.-K. Peng, S.V. Buldyrev, S. Havlin, M. Simons, H.E. Stanley, A.L. Goldberger, Phys. Rev. E 49 (1994) 1685



(1) Profile of the time series
$y_i = \sum_{k=1}^{i} (x_k - \langle x \rangle)$



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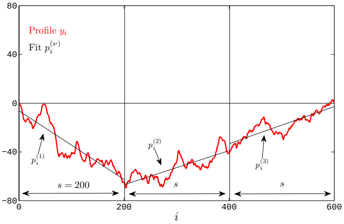
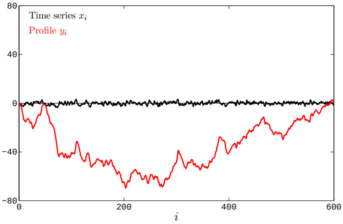


(2) Partial detrended profile
$z_i^{(\nu)} = y_i - p_i^{(\nu)}$



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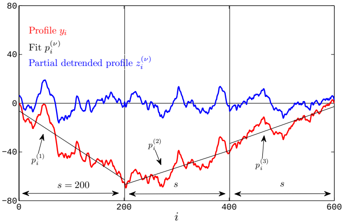

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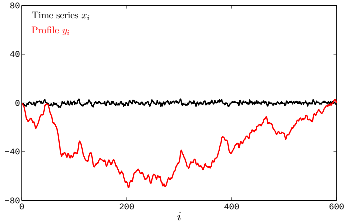

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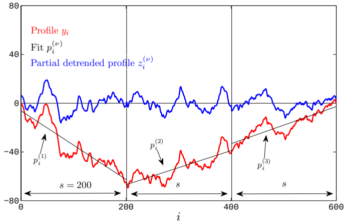

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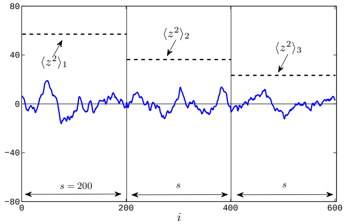
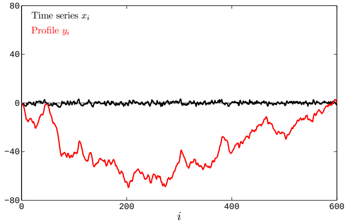


(3) Variance in segment ν
$\langle z^2 \rangle_\nu = \frac{1}{s} \sum_\nu \left(z_i^{(\nu)} \right)^2$



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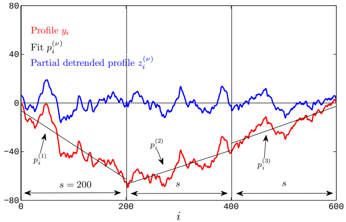

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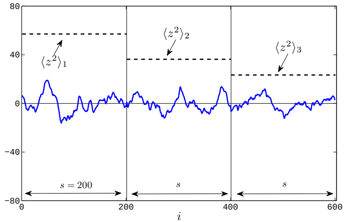
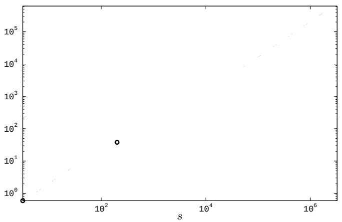
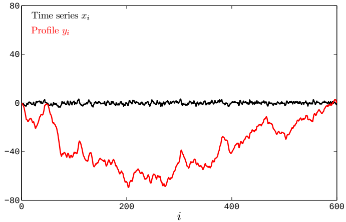


(3) Variance in segment ν
$\langle z^2 \rangle_\nu = \frac{1}{s} \sum_\nu \left(z_i^{(\nu)} \right)^2$

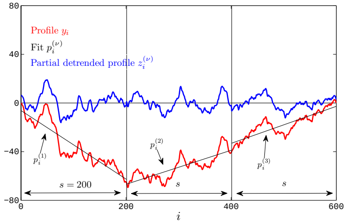


(4) Fluctuation function
$F(s) = \frac{1}{K} \sum_{\nu=1}^{K} \langle z^2 \rangle_\nu$



(1) Profile of the time series

$$y_i = \sum_{k=1}^{i}(x_k - \langle x \rangle)$$



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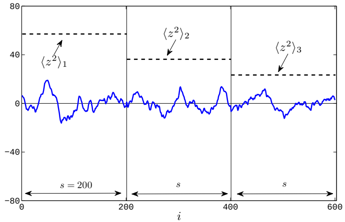
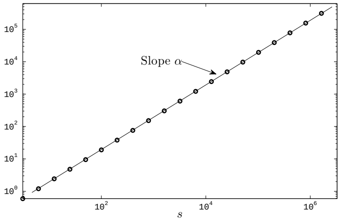
$$z_i^{(\nu)} = y_i - p_i^{(\nu)}$$



(3) Variance in segment ν

$$\langle z^2 \rangle_\nu = \frac{1}{s}\sum_\nu \left(z_i^{(\nu)}\right)^2$$



(4) Fluctuation function

$$F(s) = \frac{1}{K}\sum_{\nu=1}^{K}\langle z^2 \rangle_\nu$$

- The fluctuation function increases according to a power law

$$F(s) \sim s^{\alpha}$$

- Relationship between the fluctuation function $F(s)$ and the autocorrelation $C(s)$:
  (1) Short-range correlations

$$\alpha = \frac{1}{2}$$

  (2) Long-range correlations ($C(s) \sim s^{-\gamma}$ with $0 < \gamma < 1$)

$$\alpha = 1 - \frac{\gamma}{2} \text{ with } \frac{1}{2} < \alpha < 1$$

  (3) Other cases

$$\alpha = 1 \Rightarrow \frac{1}{f} - \text{noise}$$

$$\alpha = \frac{3}{2} \Rightarrow \text{ Brownian motion}$$

- Problem: There exists a proof of these relationship only for the case of no trends $p_i^{(\nu)} = 0$ (Fluctuation Analysis FA)
- With trends (DFA): no proof
  - Testing with artifical generated time series[1]
  - No analytical expressions for the fluctuation function, only one for the asymptotic behaviour: Fractional Brownian motion[2] ($H = \frac{1}{2} \rightarrow$ Brownian motion)

$$F(s) \sim \left(\frac{2}{2H+1} + \frac{1}{H+2} - \frac{2}{H+1} \right) s^{2H} \text{ with } 0 < H < 1$$

$$\alpha = H + 1$$

[1] J.W. Kantelhardt, E. Koscielny-Bunde, H.H.A. Rego, S. Havlin, A. Bunde, Physica A 295 (2001) 441

[2] M.S. Taqqu, V. Teverovsky, W. Willinger, Fractals 3 (1995) 785

- No proof of the DFA
- Are there any known problems?
- For instance Maraun et al.[1] applied the DFA on AR(1) processes and have shown that the fluctuation function $F(s)$ suggests long-range correlations
- The reason for this discrepancy is the undersized length of the time series

[1] D. Maraun, H.W. Rust, J. Timmer, Nonlinear Processes in Geophysics (2004) 11:495-503

Fluctuation function of the AR(1) process

- First order autoregressive process AR(1)

$$x_i = ax_{i-1} + \eta_i \text{ with } -1 < a < 1 \text{ and } \eta_i \sim \mathcal{N}(0,1)$$

  - Autocorrelation:

$$C(s) = a^s = \mathrm{e}^{-\frac{s}{s_c}}$$

    with characteristic correlation time $s_c = -\frac{1}{\ln a}$
  - DFA assumes a fluctuation parameter $\alpha = \frac{1}{2}$
- Other cases
  - $a = 0 \rightarrow$ Gaussian White noise (GWN) $\Rightarrow \alpha = \frac{1}{2}$
  - $a = 1 \rightarrow$ Random Walk (RW) $\Rightarrow \alpha = \frac{3}{2}$

- Profile

$$y_i = \sum_{k=1}^{i} x_k = \sum_{k=1}^{i} \eta_k \frac{1 - a^{1+i-k}}{1 - a}$$

- Partial detrended profile

$$z_i^{(\nu)} = y_i - (A + Bi) = \sum_{k=1}^{s} \eta_k \left(\frac{1 - a^{1+i-k}}{1 - a} \Theta(i - k) + B_{k,i} \right)$$

with $B_{k,i} = \frac{4s+2-6i}{-s^2+s} \sum_{j=k}^{s} \frac{1-a^{1+j-k}}{1-a} + 6 \frac{s+1-2i}{s^3-s} \sum_{j=k}^{s} j \frac{1-a^{1+j-k}}{1-a}$

Fluctuation function of the AR(1) process

$$F^2(s) = \frac{1}{15(a-1)^7(a+1)^2(s^4-s^2)}\Big(a^{2s}\Gamma_2(s) + a^2\Gamma_1(s) + \Gamma_0(s)\Big)$$

with the polynomials

$$\Gamma_2(s) = \gamma_{23}s^3 + \gamma_{22}s^2 + \gamma_{21}s + \gamma_{20}$$
$$\Gamma_1(s) = \gamma_{12}s^2 + \gamma_{11}s + \gamma_{10}$$
$$\Gamma_0(s) = \gamma_{05}s^5 + \gamma_{04}s^4 + \gamma_{03}s^3 + \gamma_{02}s^2 + \gamma_{01}s + \gamma_{00}$$

and the coefficients

$$\gamma_{23} = 15a^4(a-1)^3$$
$$\gamma_{22} = -60a^4(a-1)^2(a+1)$$
$$\gamma_{21} = 15a^4(5a^3 + 9a^2 - 9a - 5)$$
$$\gamma_{20} = -30a^4(a^3 + 5a^2 + 5a + 1)$$
$$\gamma_{12} = -60a^2(a-1)^3(a+1)^2$$
$$\gamma_{11} = \ldots$$

- Behaviour for large s

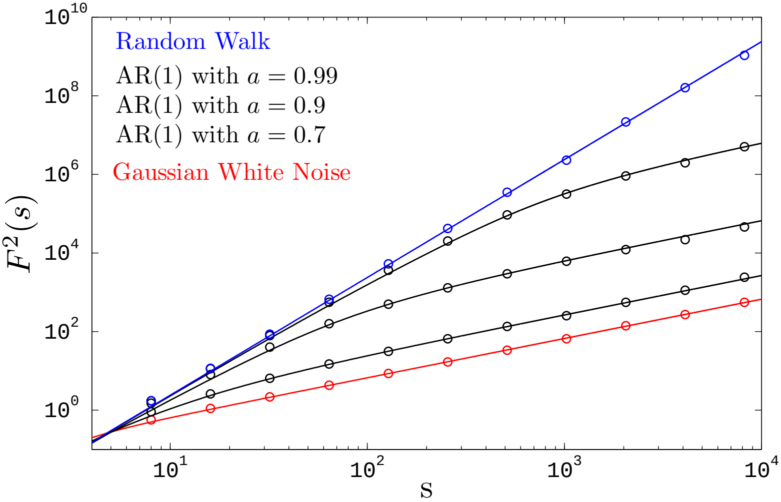
$$F^2(s) \sim \frac{1}{15(a-1)^2} s$$

- Fluctuation function of the Gaussian White Noise

$$F^2(s) = \frac{s^2 - 14}{15s} \sim \frac{1}{15} s$$

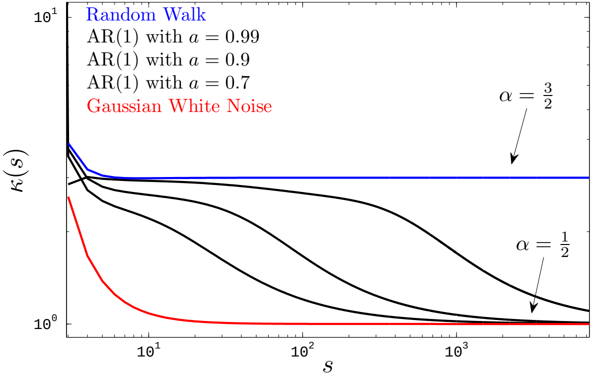
- Fluctuation function of the Random Walk

$$F^2(s) = \frac{s^4 + s^2 - 20}{420s} \sim \frac{1}{420} s^3$$



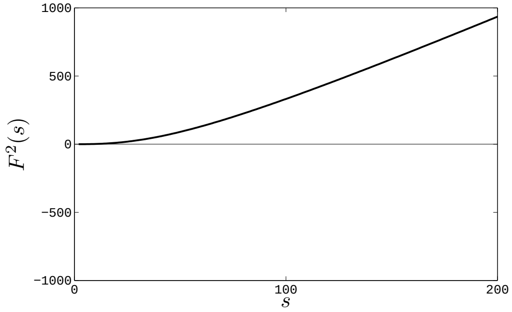
Random Walk
AR(1) with $a = 0.99$
AR(1) with $a = 0.9$
AR(1) with $a = 0.7$
Gaussian White Noise
$F^2(s)$
S
10^{0}
10^{2}
10^{4}
10^{6}
10^{8}
10^{10}
10^{1}
10^{2}
10^{3}
10^{4}

- Slope of $F^2(s)$ in the log-log plot $\kappa(s) = \frac{\mathrm{d}\ \log\ F^2(s)}{\mathrm{d}\ \log\ s} = \frac{s}{F^2(s)} \frac{\mathrm{d}F^2(s)}{\mathrm{d}s}$



- We observe for the AR(1) process:
  (1) Region with $\alpha = \frac{1}{2}$ after crossover s_x
  (2) There exists a scaling region with $\alpha > \frac{1}{2}$ before s_x

Question: How can we obtain the crossover point s_x?



Fluctuation function:

$$F^2(s) = \frac{1}{15(a-1)^7(a+1)^2(s^4-s^2)}\Big(a^{2s}\Gamma_2(s) + a^2\Gamma_1(s) + \Gamma_0(s)\Big)$$

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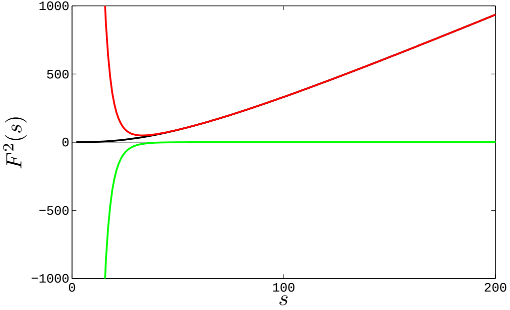
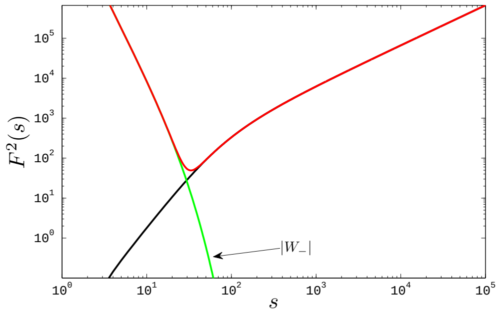


Decompostion of the fluctuation function:

$$F^2(s) = \frac{1}{15(a-1)^7(a+1)^2(s^4-s^2)}\left(a^{2s}\Gamma_2(s) + a^2\Gamma_1(s) + \Gamma_0(s)\right) = W_- + W_+$$

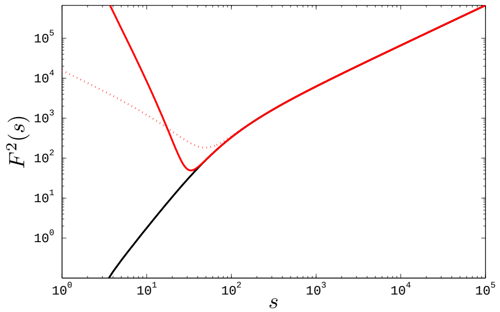
Question: How can we obtain the crossover point s_x?



Fluctuation function for sufficient large s:

$$F^2(s) \approx W_+$$

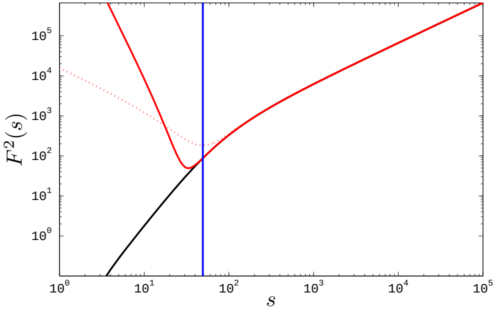
Question: How can we obtain the crossover point s_x?



Series expansion for W_+ about ∞ to order 1:

$$W_+ \approx \frac{1}{15(a-1)^2} s + \frac{a}{(a-1)^3(a+1)} + \frac{-19a^4+30a^3+68a^2+30a-4}{15(a-1)^4(a+1)^2} \frac{1}{s}$$

Question: How can we obtain the crossover point s_x?



Lower bound for the crossover:

$$s_x = \min\left(W_+{}^{1.\text{order}}\right) = \frac{\sqrt{-19a^4+30a^3+68a^2+30a-4}}{-a^2+1}$$

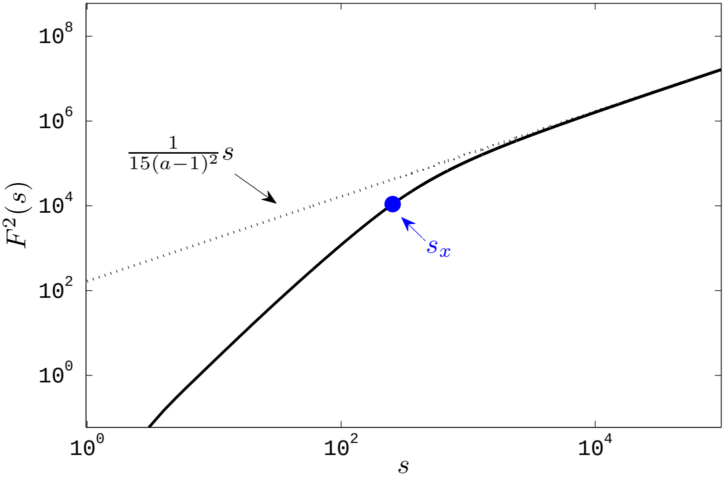
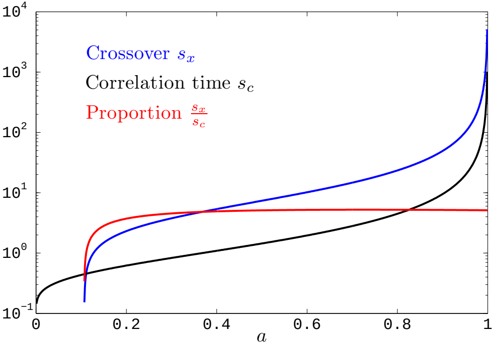


Figure : $F^2(s)$ and its asymptotic behaviour with $a = 0.98$

Comparison between the crossover s_x and the characteristic correlation time s_c (remember $C(s) = a^s = \mathrm{e}^{-\frac{s}{s_c}}$):



- The DFA needs more data for detecting s_x than a direct evaluation of s_c would need
- To determine $F(s_x)$ reliably: it takes approximately $(20 - 50)s_x$
- To determine $C(s)$: $5s_c$ is enough
- DFA is data consuming and needs about $\frac{20-50}{5}\frac{s_x}{s_c}$ more data

Summary

- Analytical expression for the fluctuation function of the AR(1) process
- Region with $\alpha > \frac{1}{2}$ (long-range) and $\alpha = \frac{1}{2}$ (short-range)
- crossover s_x
- DFA is data consuming