General theory of detrending-operation-based scaling analysis methods and applications

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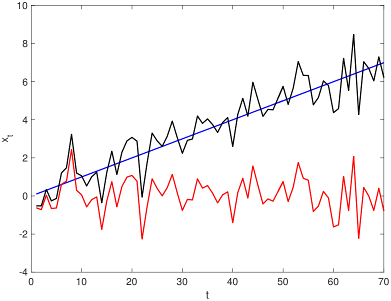
Overview

- How can we estimate the ACF for nonstationary time series?
- Given the time series x_t as

$$x_t = \eta_t + m_t$$



- Process η_t has the ACF

$$C(s) \sim \begin{cases} e^{-s/s_c} & SRC \\ s^{-\gamma} & LRC \end{cases}$$

- Indirect estimation of γ via the fluctuation function

$$F(s) \sim s^{\alpha} \text{ with } \alpha = \alpha(\gamma)$$

- Analytical framework of the relationship between C and F

Long-range correlations

- Sum of autocorrelation function (ACF) diverges

$$\sum_{s=1}^{\infty} C(s) = \sum_{s=1}^{\infty} \frac{\langle (x_t - \mu)(x_{t+s} - \mu) \rangle}{\langle (x_t - \mu)^2 \rangle} = \infty$$

- Usually described by a power law

$$C(s) \sim s^{-\gamma}$$

with correlation exponent $0 < \gamma < 1$

- The existence of LRC causes several remarkable problems in time series analysis
  - weak convergence of time averages
  - spuriously determined trends
- Detecting LRC due to trends (next page)
- First evidence of this phenomenon was found by HURST studying time series data of river flows (1951) → Hurst effect
- Two possible explanations for the Hurst effect:
  - Long-range correlations
    (first theoretical model: fractional gaussian noise by MANDELBROT 1968)
  - Short-range correlated process with additive trends
    (first investigated by BHATTACHARYA 1983)

Pitfalls in the estimation of the autocorrelation function

- Estimator of ACF has (at least) two practical problems:
  (P1) Statistical uncertainty around zero → difficult to observe power law
  (P2) Estimator only meaningful for stationary x_t
  ($\mathcal{P}2$) Possible detection of LRC due to trends

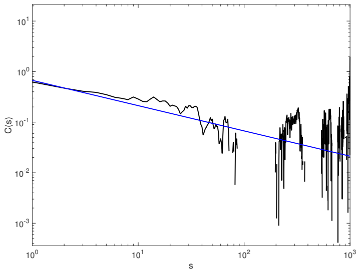


Figure: ACF of FGN with $H = 0.9$: Estimation (black) and theoretical (blue)

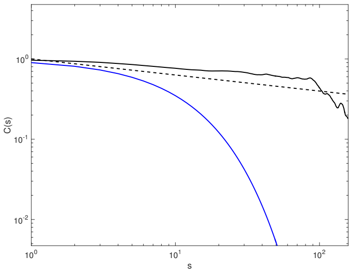


Figure: ACF of AR(1) with $a = 0.9$ and added linear trend with slope 0.02: Estimation (black), theoretical (blue) and reference line $s^{-0.2}$ (dashed)

Fluctuation function

- To overcome (P1) and (P2) we introduce the fluctuation function

$$F^2(s) = \int_0^s \mathrm{d}r\ C(r)L(r,s)$$

as an integral transform of ACF with suitable kernel L

(P1) F increases according to a power law

$$F(s) \sim s^\alpha \qquad\qquad \text{later } \alpha = \alpha(\gamma)$$

(P2) Influence of external trends should be eliminated.
In detail

$$F^2(s) = \int_0^s \int_0^{s-r'} \mathrm{d}r\mathrm{d}r'\ Cov(r,r')\tilde{L}(r,r',s)$$

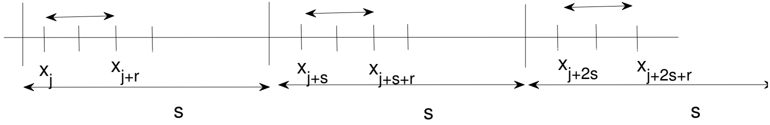
$$\xRightarrow{\text{if stationary}}$$

$$L(r,s) = \int_0^{s-r'} \mathrm{d}r'\ \tilde{L}(r,r',s)$$

→ How to estimate F for a given time series?

Estimation of the fluctuation function

- $F^2(s) = \int_0^s \mathrm{d}r\, C(r)L(r,s)$ estimated via segmentation of time axis:
Segmentation → Produce pairs $x_j x_{j+r} L(r,s)$ → Averaging



- Integrand $C(r)L(r,s)$ is estimated by compairing pairs of each segment

$$C(r)L(r,s) \approx \frac{1}{K} \sum_{\nu=1}^{K} x_{j+(\nu-1)s} x_{j+(\nu-1)s+r} L(r,s)$$

- Integral replaced by sum $\int_0^s \mathrm{d}r \to \sum_{r=0}^{s}$
- Parameter j becomes important for nonstationary series
- Famous example: detrended fluctuation analysis (DFA) → new

Detrended fluctuation analysis (DFA)

- Invented by Peng et al. (1994)
- Widely used method for the detection of long-range correlations in nonstationary time series
- Google scholar: over 10 000 results
- Applied to such diverse fields of interest as
  - DNA sequences
  - human gait
  - heart rate dynamics
  - weather records
  - economical time series
  - and many more
- **BUT: no analytical investigation, no relationship between C and F, only tested with artificial data**

- Fluctuation function is

$$\mathcal{F}^2(s) = \frac{1}{K}\sum_{\nu=1}^{K} f^2(\nu, s)$$

- Thereby DFA variances (mean-squared residual)

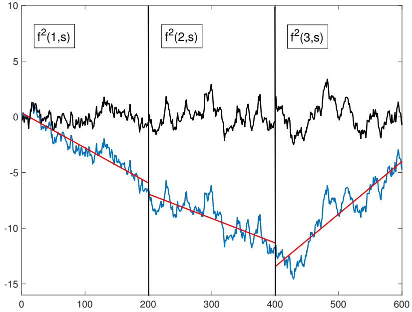
$$f^2(\nu, s) = \frac{1}{s}\sum_{t\in\nu}(y_t - p_t)^2$$

with

x_t ... time series

y_t ... profile

p_t ... fit of order q



→ **What is the connection between $\mathcal{F}$ and ACF?**

Relationship between $\mathcal{F}$ and C

(I) We assume statistical equivalence of the DFA variances

$$\langle f^2(\nu, s) \rangle = \langle f^2(\omega, s) \rangle \qquad \rightarrow \text{Crucial condition}$$

$$\Rightarrow \langle \mathcal{F}^2(s) \rangle = \frac{1}{K} \sum_{\nu=1}^{K} \langle f^2(\nu, s) \rangle = \langle f^2(1, s) \rangle = \frac{1}{s} \sum_{t=1}^{s} \langle (y_t - p_t)^2 \rangle$$

(II) Write detrended profile as sum of the original series

$$y_t - p_t = \sum_{k=1}^{s} x_k \mathcal{P}_{t,k}$$

Then

$$\langle \mathcal{F}^2(s) \rangle = \langle {x_t}^2 \rangle \sum_{r=0}^{s} C(r) L(r, s)$$

$$L(r, s) = \sum_{k=1}^{s-|r|} \sum_{t=1}^{s} \mathcal{P}_{t,k} \mathcal{P}_{t,k+|r|}$$

- $\mathcal{F}$ of DFA is a specific estimator of above fluctuation function

$$\langle \mathcal{F}^2(s) \rangle \approx F^2(s) = \int_0^s \mathrm{d}r \; C(r) L(r, s)$$

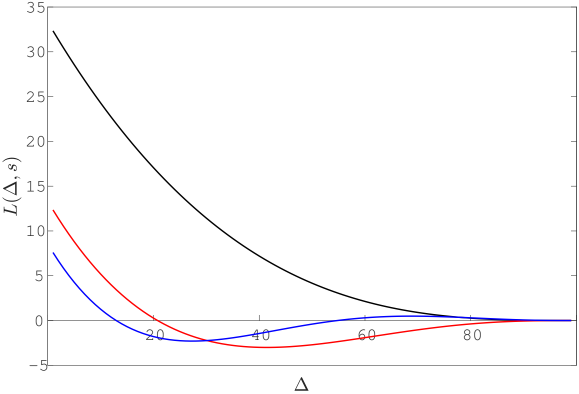


Figure: Kernel L for constant, linear and quadratic detrending, $s = 100$

Stationary series

- We can calculate analytical solutions of $\mathcal{F}$ for SRC and LRC

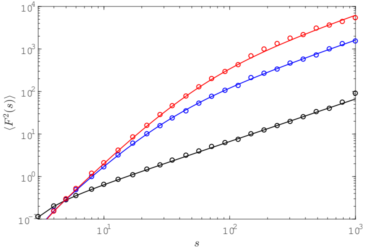


Figure: WN, AR(1) with $a = 0.8$ and $a = 0.9$

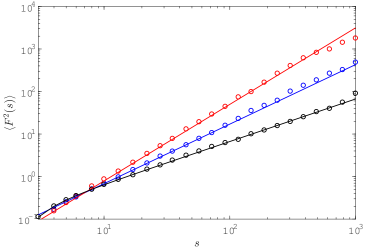


Figure: WN, ARFIMA(0,d,0) with $d = 0.2$ $(\gamma = 0.6)$ and $d = 0.4$ $(\gamma = 0.2)$

- We can derive the relationship between α and γ

$$\alpha = \begin{cases} 1/2 & \text{SRC} \\ 1 - \gamma/2 & \text{LRC } (1/2 < \alpha < 1) \end{cases}$$

- Crossover point for SRC with ACF $C(s) \sim e^{-s/s_c} = a^s$

$$s_\times = c_q \frac{e^{1/s_c}}{e^{2/s_c} - 1} = c_q \frac{a}{(1-a)(1+a)}$$

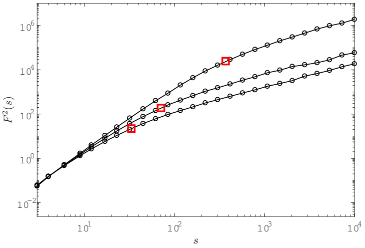


Figure: $a = 0.8$, 0.9 and 0.98

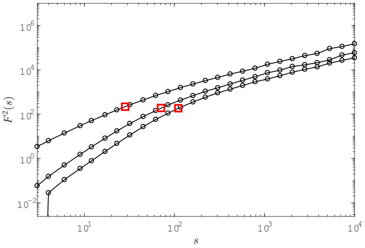


Figure: $q = 0$, 1 and 2

- DFA is data consuming (hand waving argument: about 200 more data points than direct estimation of ACF)

Nonstationary series

- Two different types of nonstationarity

$$x_t = \eta_t + m_t$$
$$\swarrow \qquad \searrow$$
intrinsic   external

- External nonstationarity: Trends $\surd$
- Intrinsic nonstationarity: FBM with $\alpha = H + 1$?
- DFA is said to be able to remove nonstationarities
- Why does it work for FBM

	FGN	FGN + mt	FBM
DFA-0	$\alpha = H$	$\alpha = 1$	$\alpha = 1$
DFA-1	$\alpha = H$	$\alpha = 1$	$\alpha = H + 1$
DFA-2	$\alpha = H$	$\alpha = H$	$\alpha = H + 1$

- Answer: from random walk to autocovariance picture

$$\langle \mathcal{F}^2(s) \rangle = \frac{1}{K} \sum_{\nu=1}^{K} \langle f^2(\nu, s) \rangle$$

$$\swarrow \qquad\qquad \searrow$$

$$\langle f^2(\nu, s) \rangle = \frac{1}{s} \sum_{t \in \nu} \langle (y_t - p_t)^2 \rangle = \sum_{i,j=1}^{s} \langle x_{i+(\nu-1)s} x_{j+(\nu-1)s} \rangle \tilde{L}(i, j, s)$$

- Detrending = Influence $\mathfrak{f}^2$ of autocovariance differences $D_{i,j}$ on variances f^2 between segments is removed such that

$$\langle f^2(\nu, s) \rangle = \langle f^2(\omega, s) \rangle$$

- Simple example of BM:
  - DFA-0: It is $\langle f^2(\nu, s) \rangle = \sum_{i,j \in \nu} \langle x_i x_j \rangle$
  - DFA-1: $\langle x_{i+s} x_{j+s} \rangle = \langle x_i x_j \rangle + \underbrace{D_{i,j}}_{=s} \Rightarrow \langle f^2(2, s) \rangle = \langle f^2(1, s) \rangle + \underbrace{\mathfrak{f}^2(\mathbb{D})}_{=0}$
- Same for FBM and trends ⇒ unified picture of detrending

Break processes

Break process

$$m_t = \begin{cases} \mu_1 & t \leq T \\ \mu_2 & t > T \end{cases}$$

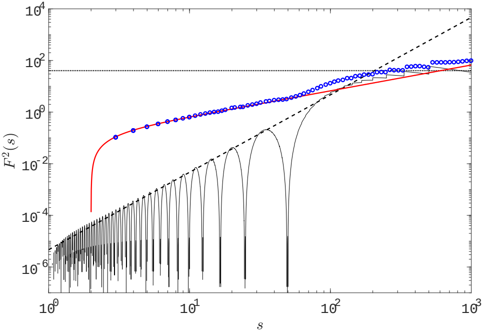


Figure: Full fluctuation function, white noise and break process