Globally Coupled Stratonovich Models: Self-consistent Theory of a Non-equilibrium Continuous Phase Transitions

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Outline

1. Introduction
2. N coupled Stratonovich models
3. Summary

1. Introduction

Stratonovich model [1]

- Langevin equation

$$\frac{d}{dt}x = f(x) + g(x) \circ \xi$$

with $f(x) = ax - x^3$ and $g(x) = \sigma x$

- Fokker-Planck-Equation (FPE)

$$\frac{\partial}{\partial t}p(x;t) = L_x p(x;t)$$

with $L_x \bullet = -\frac{\partial}{\partial x}\Big[f(x)\bullet\Big] + \frac{1}{2}\frac{\partial}{\partial x}\Big[g(x)\frac{\partial}{\partial x}\Big(g(x)\bullet\Big)\Big]$

[1] A. Schenzle, H. Brand, *Multiplicative stochastic processes in statistical physics*, Phys. Rev. A **20** 1628 (1979)

Solution of the FPE

$$p_s(x) = \begin{cases} \delta(x) & a \leq 0 \\ \frac{1}{Z}|x|^{\frac{2a}{\sigma^2}-1} e^{-\frac{x^2}{\sigma^2}} & a > 0 \end{cases}$$

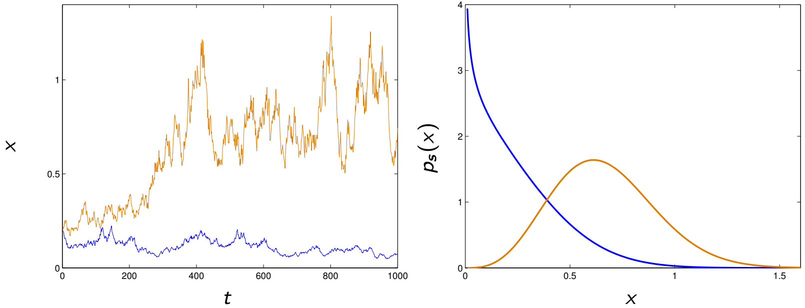

Figure: $\sigma = 0.5$, $a = 0.1$ and $a = 0.5$

Noise induced phase transition of 2. kind[1]

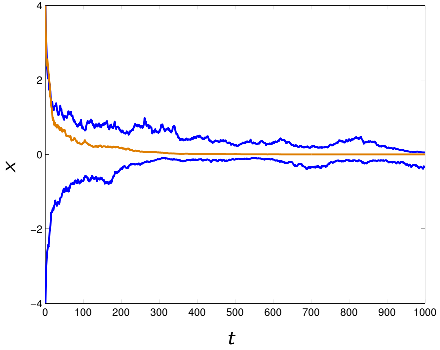

Figure: $\sigma = 0.5$, $a = -1$ and $a = 0.1$ (different inital values)

- Ergodicity breaking
- Critical behaviour of the order parameter $\langle x \rangle$ near the critical point $a_c = 0$

[1] W. Horsthemke, R. Lefever 1984 *Noise-Induced Transitions* (Berlin: Springer)

2. N coupled Stratonovich models

- N coupled Stratonovich models ($i = 1, \ldots, N$)

$$\frac{d}{dt}x_i = ax_i - {x_i}^3 + \frac{D}{N-1}\sum_{j=1}^{N}(x_j - x_i) + \sigma x_i \circ \xi_i$$

- No stationary solution of the FPE is known
  $\Rightarrow$ Approximate solution
- Here: strong coupling (D is large)
  $\Rightarrow$ Introduction of the centre of mass R and the relative coordinates r_k ($k = 2, \ldots, N$)

$$R = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$r_k = R - x_k$$

- Langevin equations transform

$$\frac{d}{dt}R = \ldots \text{ (without } D) \ldots + \sum_{i=1}^{N} g_{1i} \circ \xi_i$$

$$\frac{d}{dt}r_k = \left(a - \frac{N}{N-1}D\right) r_k + \ldots + \sum_{i=1}^{N} g_{ki} \circ \xi_i$$

- Crucial physical observation: time scale separation
  - R moves slowly
  - $r_2, \ldots, r_N$ move fast (short time scales)

2.1 Strong coupling limit $D \to \infty$[1]

Only results:

- Probability densities

$$p_s(\mathbf{r}) = \delta(\mathbf{r})$$

$$p_s(R) = \begin{cases} \delta(R) & a \leq a_c \\ \frac{1}{N_R} |R|^{\frac{2aN}{\sigma^2} + N - 2} & a > a_c \end{cases}$$

with the critical point $a_c = -\frac{\sigma^2}{2}\left(1 - \frac{1}{N}\right)$

- Order parameter $\langle R \rangle$ shows critical behaviour near a_c:

$$\langle R \rangle \sim (a - a_c)^{\beta} \text{ with } \beta = 1$$

[1] F. Senf, P.M. Altrock, U. Behn, *Nonequilibrium Phase Transitions in Finie Arrays of Globally Coupled Stratonovich Models: Strong Coupling Limit*, New J. Phys. 11, 063010 (2009).

2.1 Strong coupling limit $D \to \infty$[1]

Only results:

- Probability densities

$$p_s(\mathbf{r}) = \delta(\mathbf{r}) \Rightarrow \text{ No width.}$$

$$p_s(R) = \begin{cases} \delta(R) & a \leq a_c \\ \frac{1}{N_R} |R|^{\frac{2aN}{\sigma^2}+N-2} & a > a_c \end{cases}$$

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2.2 Finite coupled models $D \gg 1$

Two observations

(1) $\langle \mathbf{r}^2 \rangle \neq 0$

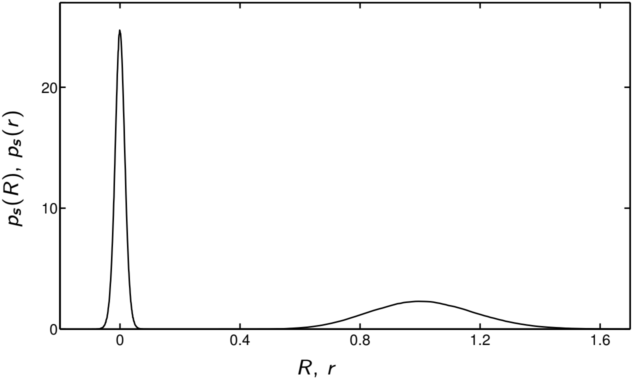

Figure: $a = 1$, $\sigma = 0.5$, $D = 130$ and $N = 2$

(2) The reduced FPE of $p(R;t)$ contains conditional moments of the relative coordinates (later with more assumptions: $\langle r_k|R\rangle$ and $\langle {r_k}^2|R\rangle$)

$\rightarrow$ The reduced FPE: Integrating the full FPE over all relative coordinates yields the reduced FPE of $p(R;t)$

$$\text{full FPE}: \frac{\partial}{\partial t} p(R, \mathbf{r}; t) = L p(R, \mathbf{r}; t)$$

$$\Big\downarrow \int d\mathbf{r}$$

$$\text{reduced FPE}: \frac{\partial}{\partial t} p(R; t) = L_R\, p(R; t)$$

with $L_R = \ldots$ contains $\langle r_k|R\rangle$ and $\langle {r_k}^2|R\rangle \ldots$

How can we obtain

$$p_s(r_k|R)?$$

(A) FPE of $p(\mathbf{r}|R;t)$

- Keep R constant in the Langevin equations for $\mathbf{r}$
  $\Rightarrow$ Corresponding FPE for $p(\mathbf{r};R,t)$
- Interpretation of $p(\mathbf{r};R,t)$:

$$\text{time scale seperation } \Rightarrow p(\mathbf{r};R,t) \approx p(\mathbf{r}|R;t)$$

- FPE for $p(\mathbf{r}|R;t)$

$$\frac{\partial}{\partial t} p(\mathbf{r}|R;t) = L_{R,\mathbf{r}} p(\mathbf{r}|R;t)$$

$\rightarrow$ Solution only for $N = 2$
$\rightarrow$ Use $p(r_k|R;t)$ instead of $p(\mathbf{r}|R;t)$

(B) Self-consistent equations

$$p_s(\mathbf{r}|R) \xrightarrow{\text{Integration}} p_s(r_k|R, \langle r_k|R\rangle, \langle {r_k}^2|R\rangle) \xrightarrow{\text{Self-cons. eq.}} p_s(r_k|R)$$

- Two ansatzes
  - (1) $p_s(\mathbf{r}|R)$ seperates
  - (2) All conditional moments are equal
- Self-consistent equations

$$\langle r_k|R\rangle = \int dr_k \ r_k p_s(r_k|R; \langle r_k|R\rangle, \langle {r_k}^2|R\rangle)$$

$$\langle {r_k}^2|R\rangle = \int dr_k \ {r_k}^2 p_s(r_k|R; \langle r_k|R\rangle, \langle {r_k}^2|R\rangle)$$

The conditional probability density

- Approximately: $p_s(r_k|R)$ is gaussian

$$p_s(r_k|R) = \frac{1}{N}\exp\left(-\frac{N^2 D}{\sigma^2(N-1)^2 R^2}{r_k}^2\right)$$

- The conditional moments are

$$\langle r_k|R\rangle = 0$$
$$\langle {r_k}^2|R\rangle = \frac{\sigma^2}{2}\frac{(N-1)^2}{N^2 D}R^2$$

The probability density $p_s(R)$

The stationary solution of the reduced FPE

$$p_s(R) = \begin{cases} \delta(R) & a \leq a_c \\ \frac{1}{N_R}|R|^{a_N} e^{-b_N R^2} & a > a_c \end{cases}$$

with the critical point $a_c = -\frac{\sigma^2}{2}\left(1 - \frac{1}{N}\right)\left(1 - \frac{(N-1)^2\sigma^2}{N^2 D}\right)$.
The coefficients are

$$a_N \approx \frac{2aN}{\sigma^2} + N - 2 - \frac{(N-1)^3(2a+\sigma^2)}{N^2 D} + \mathcal{O}(D^{-2})$$
$$b_N \approx \frac{N}{\sigma^2} + \frac{2(N-1)^3}{N^2 D} + \mathcal{O}(D^{-2})$$

(This is for $D \to \infty$)

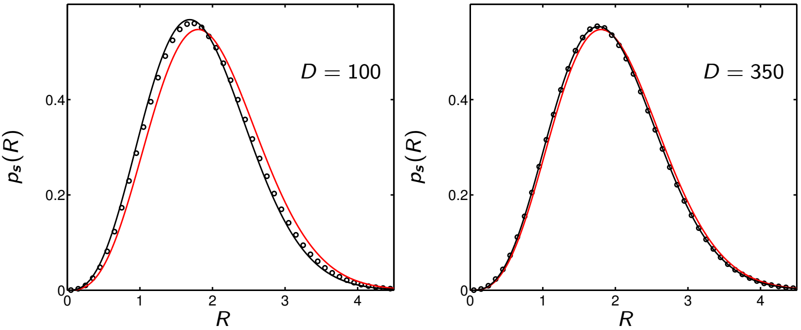

Figure: $a = 1$, $\sigma = 3$, $N = 4$ (simulat.: $\circ$, $D \to \infty$: $-$, $D \gg 1$: $-$)

The probability density $p_s(r_k)$

The stationary solution is

$$p_s(r_k) = \begin{cases} \delta(r_k) & a \leq a_c{}^{(N)} \\ \ldots r_k{}^{\frac{a_N}{2}} \mathcal{K}_{\frac{a_N}{2}}\left(\ldots |r_k|\right) & a > a_c{}^{(N)} \end{cases}$$

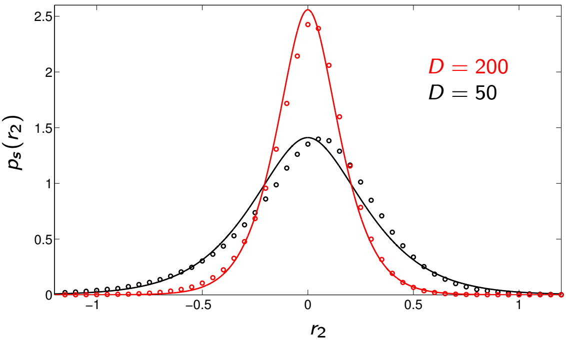

Figure: $a = 1$, $\sigma = 2.5$, $N = 5$ (simulat.: $\circ$, analytic formula: $-$)

Noise induced phase transition of 2.kind

- Second moment $\langle {r_k}^2 \rangle$ can serve as parameter

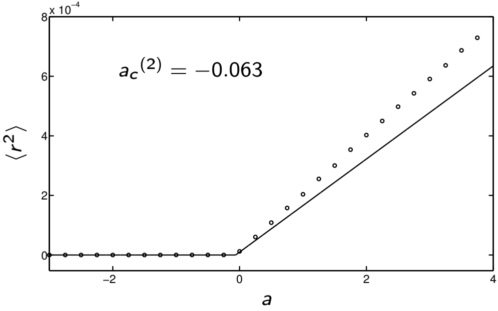

Figure: $\sigma = 0.5$, $D = 100$ and $N = 2$ (simulation: ∘, analytical formula: −)

- Power law near the critical point:

$$\langle {r_k}^2 \rangle \sim (a - a_c)^{\beta} \;\text{ with } \beta = 1$$

3. Summary

- Starting point:

$$D \to \infty \Rightarrow p_s(\mathbf{r}) = \delta(\mathbf{r})$$
$$D < \infty \Rightarrow p_s(\mathbf{r}) \text{ has finite width}$$

- Method has been presented, which evaluates
  (a) Corrections of $p_s(R)$
  (b) $p_s(r_k)$
  (c) Order parameter $\langle {r_k}^2 \rangle$